Pre-class Warm-up!!!

True or False?

The matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ has 3 eigenvectors, no two of which are scalar multiples of each other.

a. True

b. False

It has up to scalar multiple only the e-vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with e-values 2 and 3.

Do you remember last time we did a lot with the matrix $\begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$?

It has eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with eigenvalues 7, -1.

6.2 Diagonalization of matrices

New vocabulary:

- diagonalize, diagonalizable, similar

We learn:

- the connection between eigenvalues, eigenvectors and diagonalization
- how to diagonalize a matrix (when it is diagonalizable).
- Some theorems: a criterion for diagonalizability; independence of eigenvectors when eigenvalues are distinct; distinct eigenvalues implies diagonalizable.

What we don't really learn:

- why we would want to diagonalize matrices

Definition. Square matrices A and B are similar if there is an invertible matrix P so that

$$B = P^{-1} A P$$

A square matrix A is diagonalizable if it is similar to a diagonal matrix. So:

$D = P^{-1} A P$ is diagonal for some invertible P.

Definition. Square matrices A and B are similar if there is an invertible n x n matrix P so that $P^{-1}AP = B$

A square matrix A is diagonalizable if it is similar to a diagonal matrix.

Example: $A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$

Try the matrix $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ so $P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Calculate

$$P^{-1}AP = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 7 & -1 \\ 7 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 14 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & -1 \end{bmatrix}$$

is diagonal.

Wow!! The columns of P are the e-vectors of A and the diagonal entries of $\begin{bmatrix} 7 & 0 \\ 0 & -1 \end{bmatrix}$ are the corresponding e-values.

Theorem. Let A, P, D be n x n matrices with P invertible and D diagonal.
Then P^{-1}AP = D if and only if the columns of P are eigenvectors for A with eigenvalues the diagonal entries in D.

Proof. $P^{-1}AP = D \iff AP = PD$

Write the columns of P as $v_1, \dots, v_n$

$P = [v_1 | v_2 | \cdots | v_n]$. Write $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$AP = PD$ means: for each i, $Av_i = i^{th}$ col of AP

$= \lambda_i v_i = i^{th}$ column of PD.

e.g. $\begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ 7 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 7 & \\ & -1 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ 7 & 1 \end{bmatrix}$

$\iff$ for each i, $Av_i = \lambda_i v_i$

$\iff$ for each i, v_i is an e-vector of A with e-value λ_i $\square$

Theorem 1. An n x n matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

Proof. A has n linearly independent e-vectors

$\iff$ there is an invertible matrix P with columns that are e-vectors of A

$\iff$ there is invertible P with $P^{-1}AP = D$ is diagonal.

$\iff$ A is diagonalizable.

Example. The matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.

Proof. We show A does not have 2 independent e-vectors. Find e-values:

Characteristic poly: $\det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}$

$= (1-\lambda)^2$ Roots: 1 (twice).

To find e-vectors: find $\mathrm{Null}(A - \lambda I)$

$= \mathrm{Null} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

1 free variable. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a basis for the nullspace. There is one e-vector up to scalar multiple. A is not diagonalizable.

Also $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 17 \\ 0 & 2 \end{bmatrix}$ are not diagonalizable.

Like 6.2 questions 1-28

Find whether or not the following matrices are diagonalizable. If so, find P so that $P^{-1}AP = D$ is diagonal.

1. $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ 2. $A = \begin{bmatrix} -5 & -12 \\ 3 & 7 \end{bmatrix}$

Solution: Char poly $(1-\lambda)(3-\lambda)$. Two e-values

1: find nullspace of $A - I = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$.

It has basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

3. $\mathrm{Null} \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}$ has basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Take $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

Like 6.2 questions 1-28

Find whether or not the following matrices are diagonalizable. If so, find P so that P^{-1}AP = D is diagonal.

1. $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ 2. $A = \begin{bmatrix} -5 & -12 \\ 3 & 7 \end{bmatrix}$

Two more matrices:

3. $A = \begin{bmatrix} -5 & -14 \\ 3 & 8 \end{bmatrix}$ 4. $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

Solution 2. Char. poly $= \det \begin{bmatrix} -5-\lambda & -12 \\ 3 & 7-\lambda \end{bmatrix}$

$= \lambda^2 + 5\lambda - 7\lambda - 35 + 36 = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$

$\lambda = 1$ is the only root, twice.

Find e-vectors: Null $A - I$ = Null $\begin{bmatrix} -6 & -12 \\ 3 & 6 \end{bmatrix}$

Echelon form $\begin{bmatrix} -6 & -12 \\ 0 & 0 \end{bmatrix}$. Basis for nullspace

$\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. There is only on e-vector up to scalar multiple. A is not diagonalizable

Question: How many distinct eigenvalues does matrix 4. have? If you get to it: is it diagonalizable?

a. 0

b. 1

c. 2

Theorem 2. If A has eigenvectors v_1, … v_k associated to distinct eigenvalues, then v_1, … , v_k are independent.

Proof. Let the e-values be $\lambda_1, \dots, \lambda_k$

so $Av_i = \lambda_i v_i$ for each i,

Suppose $v_1, \dots, v_k$ were dependent

Let $c_1 v_1 + \dots + c_k v_k = 0$ be a non-zero dependence relation. Reorder the v_i and let $c_1 v_1 + \dots + c_r v_r$ be a shortest such relation. Apply the matrix $A - \lambda_r I$,

$$(A - \lambda_r I)\, v_i = Av_i - \lambda_r v_i = \lambda_i v_i - \lambda_r v_i$$

$$\begin{aligned} &= 0 \text{ if } i = r \\ &\neq 0 \text{ if } i \neq r, \end{aligned}$$

We get a shorter relation if $r > 1$

This is a contradiction. The $v_1, \dots, v_k$ are independent.

Theorem 3. If the n x n matrix A has n distinct eigenvalues, it is diagonalizable.

Page 354 Question 32
Show that if $n \times n$ matrices A and B are similar, then they have the same characteristic equation, and therefore have the same eigenvalues.

Page 354 Question 29
Prove: if the matrices A and B are similar and the matrices B and C are similar, then the matrices A and C are similar.